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Econometric Model Used in the Portfolio Optimization over Several Periods

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Abstract

The classic problem of optimizing a portfolio can be extended to a multi-stage programming problem. The purpose of the multi-period portfolio optimization problem is to determine the optimal portfolio for a certain finite time horizon.
In a multi-period model in which investors are allowed to change the composition of the portfolio, it is essential to take into account trading costs, a solution in this regard being the use of tree-type scenarios.
The study undertaken by the authors considered the construction of a portfolio optimization model in case there is a certain constraint on returns. ARMA type processes were used to model the conditional mean equation.

Keywords: *model; variables; estimation, portfolios; capital market; investments.*

JEL Classification: C13; G11.

Introduction

The study conducted by the authors, based on financial market data, highlighted a number of issues related to portfolio investments over several periods of time. In practice, these are alternative periods, choosing the one that is applied to lead to results.

The econometric model can be used favorably to optimize the placement of asset portfolios on the capital markets in various situations and especially in different time periods. Only a proper market study provides the arguments and conclusions that can underpin the decisions that are made. The econometric model is one that is based on statistical variables specific to the portfolios and the market in which it is invested.

The study underlying this article analyzes in depth the main statistical-econometric models that can be adapted and used in optimizing the placement of asset portfolios on the capital market, during the periods that are most profitable.

In presenting the research topic, the main stages are covered, presenting the main statistical-econometric models with emphasis on identifying possibilities to adapt to the specifics of the capital market, so that the model built to ensure the optimization of the portfolio considered. In determining the return of a placed portfolio of assets, an important role is played by the period we have in mind.

Literature Review

Angelelli, Mansini and Speranza (2008) analyzed two linear programming models to solve the portfolio selection problem. Anghel (2020) studied the use of ARIMA and VAR models in economic analyzes. Anghel (2013) analyzed the main portfolio management models. Anghelache, Anghel and Iacob (2020) used econometric tools in the analysis of the capital market. Anghelache and Anghelache (2014) addressed a number of issues related to investments in financial instrument portfolios. Audrino (2005) proposed a non-parametric probability estimator. A similar topic was studied by Franco and Zakoian (2004). Barone-Adesi, Engle and Mancini (2008), and Engle (2001) analyzed the main aspects of using GARCH models. The VAR model was analyzed by Juselius (2006) as well as by Mancini and Trojani (2011). Koonce, Lipe and McAnally (2005) studied elements of financial instrument risk. Wooldrige (2006) presented the main econometric methods and models used in economic analyzes. Zumbach (2010) studied various aspects of volatility forecasting.

Methodology, Data, Discussions, Results

If we consider that the initial investment was made at time t = 0, the portfolio could be restructured at times t = 1, 2, ..., T-1 and redeemed at the end of the period (t = T), according to the scenario presented in Figure 1.

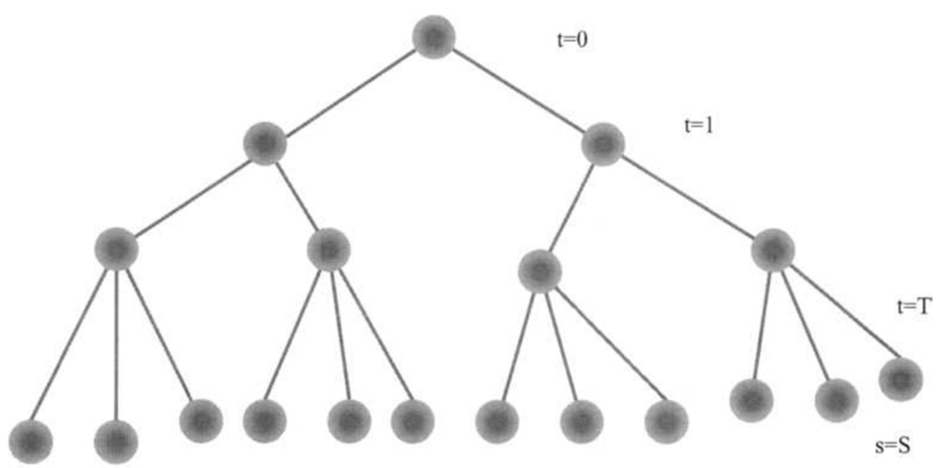

Fig. 1. Tree scenario

To solve the problem of portfolio optimization over several periods, it is necessary to generate future prices at different time intervals for the assets in the portfolio. This can be done using tree scenarios. Future prices are approximated by a discrete set of scenarios, or sequence of events.

Given the historical developments for an event up to a certain point, the uncertainty in the next period is characterized by a finite number of possible results for the next observation.

Let N be a set of nodes in the tree scenario. Each node of the scenario tree is denoted by $e = (s,t)$, where s is a scenario and the time period t specific to a particular node in this way. The root node,, $N_0 = (s,0)$, in the tree scenario, represents the current time period, and the asset prices at the root node are easily observable. The event before event e is denoted a $a(e) = s,t\text{-}1$.

A probability P_e is attached to each branch. The leaves in the tree represent the value of asset prices after T periods of time, where T is the depth of the tree scenario. The set of unique paths from root to leaf nodes represents the set of scenarios, and the cumulative probability on each branch in the scenario represents the probability of this scenario. Therefore, the scenario set corresponds to the leaf nodes of the N_T tree scenario.

Several scenarios have been generated in the literature. A portfolio optimization tree scenario starts from the assumption that the chosen scenario must meet certain conditions. It is stated that it is not enough just to comply with certain statistical conditions, but to introduce arbitration constraints in order to get the most realistic results possible.

An approach can also be described by simulations and random clusters of the tree scenario. Each node of the tree scenario contains a cluster of scenarios, which becomes a center of gravity. The final shaft consists of the centers of gravity of each node and the probabilities attached to each branch. A random clustering algorithm is used for simulations.

Taking into account trading costs, the balance of a portfolio can be calculated starting from the equation:

$$r_t^s w_{t-1}^s + (1-c)\overline{\delta}_t^s - (1+c)\underline{\delta_t^s} = w_t^s \tag{1}$$

where:

w_t^s is a vector that contains the balance of assets at a time t and in the case of a certain scenario s;

r_t^s is a vector of asset returns in the case of scenario s and at time t;

$\underline{\delta_t^s}$ is a vector of shares sold at time t in the case of scenario s;

$\overline{\delta}_t^s$ is a vector of the shares bought at time t;

c is a vector at transaction costs for each asset.

- After the method of generating the future price trajectory has been represented, the model for describing the returns over several periods can also be specified. It is possible to build a model that optimizes a portfolio in the event that a certain constraint on returns is required. Thus, we start from the minimum condition:

$$\min_{w,\varsigma}\left[\varsigma + \frac{1}{(1-\alpha)}\sum_{s=1}^{S} p_s z_s\right] \tag{2}$$

Provided that:

$$P + (1-c)\overline{\delta}_0 - (1+c)\underline{\delta_0} = w_0 \tag{3}$$

$$1'\overline{\delta}_0 - 1'\underline{\delta_0} = 1 - 1'P \tag{4}$$

$$r_t^s w_{t-1}^s + (1-c)\overline{\delta}_t^s - (1+c)\underline{\delta_t^s} = w_t^s \tag{5}$$

$$t = 1,2,\dots,T$$

$$1'\overline{\delta}_0 - 1'\underline{\delta_0} = 0 \tag{6}$$

$$t = 1,2,\dots,T$$

$$\sum_{s=1}^{S} p_s w_T^s \geq R \quad (7)$$

$$\frac{|w_t^s|_\infty}{1' w_t^s} \leq \nu \quad (8)$$

$$z_s \geq W_0 - W_0(1' w_T^s) - \varsigma \geq 0 \quad (9)$$

$$s = 1,2, \dots, S$$

$$0 \leq \overline{\delta}_t^s \leq \overline{\Delta}_t^s \quad (10)$$

$$s = 1,2, \dots, S \text{ și } t = 0,1, \dots, T$$

$$0 \leq \underline{\delta}_t^s \leq \underline{\Delta}_t^s \quad (11)$$

$$s = 1,2, \dots, S \text{ și } t = 0,1, \dots, T$$

$$w_t^L \leq w_t^s \leq w_t^U \quad (12)$$

$$s = 1,2, \dots, S \text{ și } t = 0,1, \dots, T$$

where:

p_s is the probability that a scenario will happen;

W_0 is the initial fortune;

w_t^s is a vector containing the balance of assets at a time t and in the case of a certain scenario *s*;

r_t^s is a vector of asset returns in the case of scenario s and at time *t*;

$\underline{\delta}_t^s$ is a vector of shares sold at time *t* in the case of scenario *s*;

$\overline{\delta}_t^s$ is a vector of the shares bought at time *t*;

c is a vector of transaction costs for each asset;

P is a vector that represents the initial portfolio;

ν is the maximum proportion of an asset that can be held in the portfolio;

1 is the vector (1,1,1,1...).

Similar to the one-period optimization aspect, the constraints (2) and (9) aim at minimizing the CVaR for the portfolio. But in the multi-period situation, instead of a single-period scenario, a multi-period scenario appears. The constraint (7) ensures that the model produces a return greater than or equal to the expected minimum return on the portfolio. The constraint (8) ensures that an asset cannot be held in a proportion greater than the established fraction. Constraints (9), (10), (11) describe the vectors that represent the weights in the portfolio, the purchases and sales of assets.

The conditional heteroskedasticity can be modeled using the ARCH (AutoRegressive Conditional Heteroskedasticity) process. This process uses the residues from the conditional mean equation in the conditional variance equation. Following the analyzes, it was concluded that in order to capture the dynamics of the conditioned variance, an ARCH process with a very large number of parameters is necessary, estimating this model being difficult. Thus, the GARCH (Generalized AutoRegressive Conditional Heteroskedasticity) process was introduced.

To take into account the asymmetry effect, the two models presented were extended, appearing the EGARCH model (Exponential GARCH), the TARCH model (Threshold ARCH) and the AP ARCH model (Asymmetric Power ARCH). In order to obtain a theoretical distribution similar to the empirical one, GARCH models have been developed that use leptokurtotic distributions. At the same time, a GARCH model using the asymmetric Student distribution can be used.

Another aspect analyzed is that of the persistent volatility of financial data. It can be concluded that the volatility of the financial data series is persistent. Long-memory GARCH processes have been proposed to model persistence in volatility. The most used models that fall into this category are the FIGARCH (Fractionally Integrated GARCH) models.

We will consider y_t the analyzed data series. Assuming that:

$$y_t = E_{t-1}(y_t) + \varepsilon_t \tag{13}$$

where:
E_{t-1} represents the average conditioned by the information at time *t*;
ε_t represents the unpredictable part with the property that $E(\varepsilon_t) = 0, E(\varepsilon_t \varepsilon_s) = 0,\ \forall t \neq s$.

- Models used for the conditional mean equation and for the conditional variance equation

The GARCH models start from the assumption that the conditioned variance is not constant over time. Thus we have:

$$\varepsilon_t = z_t \sigma_t \tag{14}$$

where:
z_t (not necessarily normal) with $E(z_t) = 0$,$Var(z_t) = 1$;
σ_t is the conditional variance.

In order to elaborate the equation of the conditioned mean, the most common ways to model it are the ARMA processes (m, n). Thus we obtain the relation:

$$\psi(L)(y_t - \mu) = \Theta(L)\varepsilon_t \tag{15}$$

where:
$\psi(L) = 1 - \sum_{i=1}^{m} \psi_i L^i$;
μ is a constant.

These processes can be extended to take into account exogenous variables. Thus, the ARMAX (m, n) and ARIMA (m, l, n) models were developed.

In the case of ARMAX models (m, n), the equation of the conditioned mean is of the form:

$$\psi(L)(y_t - \mu_t) = \Theta(L)\varepsilon_t \tag{16}$$

In the case of ARIMA type processes (m, l, n), used in the case of non-stationary series, the equation of the conditioned mean is:

$$\psi(L)(1 - L)(y_t - \mu_t) = \Theta(L)\varepsilon_t \tag{17}$$

The ARFIMA model (m, d, n) is part of the series of models with fractional integration and is used to capture the long memory property present in some series of financial data (the presence of autocorrelation between observations separated by long periods of time). Thus, the equation of the conditioned mean has the following form:

$$\psi(L)(1 - L)^d(y_t - \mu_t) = \Theta(L)\varepsilon_t \tag{18}$$

where: $0 < d < 1$ represents the fractional integration coefficient, and

$$(1 - L)^d = \sum_{k=0}^{\infty} \frac{\Gamma(d+1)}{\Gamma(k+1)\Gamma(d-k+1)} L^k = 1 - \sum_{k=1}^{\infty} x_k L^k \tag{19}$$

This model captures the long memory property through the operator $(1-L)^d$, which is actually an infinite sum of data series lags. In general, in practice, this infinite amount must be truncated.

Regarding the conditional variance equation, the ARCH (q) model is the first process used in finance to model the phenomenon of volatility clustering. This model starts from the hypothesis that:

$$\sigma_t^2 = w + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 \quad (20)$$

where: $w \geq 0 \; and \; \alpha_i \geq 0$.

As can be seen the conditioned variance depends on the square of the previous residues in the equation of the conditioned mean. According to the model, if at time *t-i* there was a large change in the price of the asset, at time *t* the conditional variance will be large. As a result, the probability that there will be a large variation in profitability increases at time *t*. Thus, there will be periods of high volatility and periods of low volatility.

If $w > 0 \; and \; \sum_{i=1}^{q} \alpha_i < 1$ the ARCH process (q) is stationary with unconditional variance $\sigma^2 = E(\varepsilon_t^2) = \frac{w}{1-\sum_{i=1}^{q} \alpha_i}$.

A very common process used in finance is the GARCH process (p, q). The process is meant to capture the dynamics of conditional variance using a small number of parameters. Thus, the equation of the conditioned variance is of the form:

$$\sigma_t^2 = w + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 \quad (21)$$

where: $w \geq 0 \; and \; \alpha_i \geq 0, i = 1 \dots q \; and \; \beta_i \geq 0, i = 1 \dots p$

The equation (20) can also be written in the form:

$$\sigma_t^2 = w + \alpha(L)\varepsilon_t^2 + \beta(L)\sigma_t^2 \quad (22)$$

where: $\alpha(L) = \sum_{i=1}^{q} \alpha_i L^i, \;\; \beta(L) = \sum_{i=1}^{p} \beta_i L^i$

It follows from (22) that:

$$\sigma_t^2 = w[1-\beta(L)]^{-1} + \alpha(L)[1-\beta(L)]^{-1}\varepsilon_t^2 \quad (23)$$

which is an ARCH(∞) process.

Therefore, it can be stated that the GARCH process (p, q) is equivalent to an ARCH (∞) process, but has a small number of parameters that must be estimated.

If $w > 0$ and $\sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \beta_i < 1$, the GARCH(p,q) process is stationary with unconditional variance $\sigma^2 = E(\varepsilon_t^2) = \frac{w}{1-\sum_{i=1}^{q} \alpha_i - \sum_{i=1}^{p} \beta_i}$.

Often in the case of financial data series it was observed that $\sum_{i=1}^{q} \alpha_i - \sum_{i=1}^{p} \beta_i \geq 1$. Thus, the influence of σ^2 on σ_{t+h}^2 is preserved for high values of *h*, observing a certain persistence in volatility. This persistence can be modeled using the IGARCH(p,q) process.

The IGARCH(p,q) model assumes that $\sum_{i=1}^{q} \alpha_i - \sum_{i=1}^{p} \beta_i = 1$. In this case, the polynomial $1 - \alpha(L) + \beta(L)$ has a root equal to 1.

If we consider: $\varphi(L) = [1 - \alpha(L) + \beta(L)](1-L)^{-1}$.

It follows from (22) that:

$$[1-\alpha(L)+\beta(L)]\varepsilon_t^2 = w + [1-\beta(L)](\varepsilon_t^2-\sigma_t^2)$$

Resulting:

$$\sigma_t^2 = w[1-\beta(L)]^{-1} + \{1-\varphi(L)(1-L)[1-\beta(L)]^{-1}\}\varepsilon_t^2 \quad (24)$$

According to the TARCH model (p, q) the equation of the conditioned variance has the following form:

$$\sigma_t^2 = w + \sum_{i=1}^{q}(\alpha_i\varepsilon_{t-i}^2 + \gamma_i d_{t-i}\varepsilon_{t-i}^2) + \sum_{i=1}^{p}\beta_i\sigma_{t-i}^2 \quad (25)$$

where $d_t = \begin{cases} 1, \varepsilon_t < 0 \\ 0, \varepsilon_t \geq 0 \end{cases}$

The model takes into account the asymmetry effect present in some financial series. Due to the dummy variable d_t a decrease has a different impact than an increase. With the help of this model the existence of the asymmetry effect can be tested (coefficients γ_i significantly different from zero).

This, another model used by economists, is the APARCH(p, q) model. In the case of the APARCH(p, q) model there is no equation for the conditioned variance (σ_t^2), but for a power function whose exponent is estimated in the model. So:

$$\sigma_t^\delta = w + \sum_{i=1}^{q}\alpha_i(|\varepsilon_{t-i}| - \gamma_i\varepsilon_{t-i})^\delta + \sum_{i=1}^{p}\beta_i\sigma_{t-i}^\delta \quad (26)$$

Where $\delta > 0$ and $-1 < \gamma_i < 1$.

By means of the parameters γ_i,the APARCH(p,q) model takes into account the asymmetry effect. If $\sum_{i=1}^{q}\alpha_i E(|z| - \gamma_i z)^\delta + \sum_{i=1}^{p}\beta_i < 1$ the model is stationary and

$$E(\varepsilon_t^\delta) = \frac{w}{1-\sum_{i=1}^{q}\alpha_i E(|z|-\gamma_i z)^\delta + \sum_{i=1}^{p}\beta_i} \quad (27)$$

The value of $k_i = \sum_{i=1}^{q}\alpha_i E(|z| - \gamma_i z)^\delta$ d depends on the shape of the distribution chosen to model z_t from equation (14). Thus, in the case of normal distribution we have:

$$k_i = \frac{(1+\gamma_i)^\delta + (1-\gamma_i)^\delta}{\sqrt{2\pi}} 2^{\frac{\delta-1}{2}}\Gamma\left(\frac{\delta+1}{2}\right) \quad (28)$$

In the case of the Student distribution with ν degrees of freedom, the value of k_i is given by:

$$k_i = \frac{(1+\gamma_i)^\delta + (1-\gamma_i)^\delta}{\sqrt{(\nu-2)\pi}} \cdot \frac{\Gamma\left(\frac{\delta+1}{2}\right)\Gamma\left(\frac{\nu-\delta}{2}\right)(\nu-2)^{\frac{\delta+1}{2}}}{2\Gamma\left(\frac{\nu}{2}\right)} \quad (29)$$

In the case of the asymmetric Student distribution with ν degrees of freedom and asymmetry coefficient ξ the value of k_i has the following form:

$$k_i = \left[\xi^{-(\delta+1)}(1+\gamma_i)^{\delta} + \xi^{\delta+1}(1-\gamma_i)^{\delta}\right]\frac{\Gamma(\frac{\delta+1}{2})\Gamma(\frac{\nu-\delta}{2})(\nu-2)^{\frac{\delta+1}{2}}}{(\xi+\frac{1}{\xi})\sqrt{(\nu-2)\pi}\Gamma(\frac{\nu}{2})} \quad (30)$$

Another form of the equation for the conditioned variance is represented by the EGARCH model (p, q) and being of the form:

$$\ln\sigma_t^2 = w + \alpha(L)\left[\gamma_1\frac{\varepsilon_t}{\sigma_t} + \gamma_2\left(\left|\frac{\varepsilon_t}{\sigma_t}\right| - E\left|\frac{\varepsilon_t}{\sigma_t}\right|\right)\right] + \beta(L)\ln\sigma_t^2 \quad (31)$$

where $\alpha(L) = \sum_{i=1}^{q}\alpha_i L^i$, $\beta(L) = \sum_{i=1}^{p}\beta_i L^i$, and $E|z_t|$ depends on the shape of the distribution. Thus, the normal distribution $E|z_t|$ is presented in the form of:

$$E|z_t| = \sqrt{\frac{2}{\pi}}$$

In the case of the Student distribution with ν degrees of freedom, $E|z_t|$ to miss:

$$E|z_t| = \frac{\Gamma(\frac{1+\nu}{2})\sqrt{\nu-2}}{1+\Gamma(\frac{\nu}{2})(\nu-1)\sqrt{\pi}} \quad (32)$$

In the case of the asymmetric Student distribution with ν degrees of freedom and asymmetry coefficient ξ, $E|z_t|$ is given by the relation:

$$E|z_t| = \frac{4\xi^2\Gamma(\frac{1+\nu}{2})\sqrt{\nu-2}}{\xi+\frac{1}{\xi}\Gamma(\frac{\nu}{2})(\nu-1)\sqrt{\pi}} \quad (33)$$

Another way to model persistence in volatility is the FIGARCH model (p, d, q).

For a GARCH model we have:

$$[1-\alpha(L)-\beta(L)]\varepsilon_t^2 = w + [1-\beta(L)](\varepsilon_t^2 - \sigma_t^2) \quad (34)$$

The FIGARCH model assumes that $1-\alpha(L)-\beta(L) = \varphi(L)(1-L)^d$ where $0 \le d \le 1$.

Thus, it follows that:

$$\varphi(L)(1-L)^d\varepsilon_t^2 = w + [1-\beta(L)](\varepsilon_t^2 - \sigma_t^2) \quad (35)$$

and the equation of variance becomes:

$$\sigma_t^2 = w[1-\beta(L)]^{-1} + \{1-\varphi(L)(1-L)^d[1-\beta(L)]^{-1}\}\varepsilon_t^2 \quad (36)$$

GARCH models are estimated by the maximum likelihood method. The likelihood function depends on the type of density considered for z_t in equation (14):

In this condition of the normal distribution, the likelihood function is:

$$L = -\frac{1}{2}\sum_{t=1}^{T}(\ln 2\pi + \ln\sigma_t^2 + z_t^2) \quad (37)$$

In the case of the Student distribution with ν degrees of freedom, this is given by the following form:

$$L = T\left\{\ln\left[\Gamma\left(\frac{\nu+1}{2}\right)\right] - \ln\left[\Gamma\left(\frac{\nu}{2}\right)\right] - \frac{1}{2}\ln[\pi(\nu-2)]\right\} - \frac{1}{2}\sum_{t=1}^{T}\left\{\ln\sigma_t^2 + (1+\nu)\ln\left[1+\frac{z_t^2}{(\nu-2)}\right]\right\} \quad (38)$$

Regarding the form of the likelihood function in the case of the asymmetric Student distribution with ν degrees of freedom and asymmetry coefficient ξ, this is:

$$L = T\left\{\ln\left[\Gamma\left(\frac{\nu+1}{2}\right)\right] - \ln\left[\Gamma\left(\frac{\nu}{2}\right)\right] - \frac{1}{2}\ln[\pi(\nu-2)] + \ln\left(\frac{2}{\xi+\frac{1}{\xi}}\right) + \ln s\right\} - \frac{1}{2}\sum_{t=1}^{T}\left\{\ln\sigma_t^2 + (1+\nu)\ln\left[1+\frac{(sz_t+m)^2}{(\nu-2)}\xi^{-2d_t}\right]\right\} \quad (39)$$

where $d_t = \begin{cases} 1, z_t \geq -\frac{m}{s} \\ -1, z_t < -\frac{m}{s} \end{cases}, m = \frac{\Gamma\left(\frac{\nu+1}{2}\right)\sqrt{\nu-2}}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi}}\left(\xi+\frac{1}{\xi}\right), s = \sqrt{\left(\xi^2+\frac{1}{\xi^2}-1\right)-m^2}$

If the conditioned mean is modeled using an ARFIMA process (m, d, n) (18), the forecast for *h* periods has the following form:

$$\hat{y}_{t+h|t} = E_t(y_{t+h}) = \left[\hat{\mu}_{t+h|t} + \sum_{i=1}^{\infty}\chi_i(\hat{y}_{t+h-i|t} - \hat{\mu}_{t+h|t})\right] + \sum_{i=1}^{m}\hat{\psi}_i\left\{\hat{y}_{t+h-i|t} - \left[\hat{\mu}_{t+h|t} + \sum_{k=1}^{\infty}\chi_k(\hat{y}_{t+h-k-i|t} - \hat{\mu}_{t+h|t})\right]\right\} + \sum_{i=1}^{n}\Theta_i\hat{\varepsilon}_{t+h-i|t} \quad (40)$$

where $\hat{\mu}_{t+h|t} = E_t(\mu_{t+h}), \hat{y}_{t+i|t} = y_{t+i}, i < 0, \hat{\varepsilon}_{t+i|t} = \varepsilon_{t+i}, i < 0, \hat{\varepsilon}_{t+i|t} = 0, i \geq 0$

Below are the relationships that allow us to obtain forecasts for h periods for the most used models to calculate the conditional average. Therefore, in the case of the AR model (1) we obtain:

$$\hat{y}_{t+h|t} = E_t(y_{t+h}) = \hat{\mu}_{t+h|t} + \psi_1(\hat{y}_{t+h-1|t} - \hat{\mu}_{t+h|t}) \quad (41)$$

For the ARMA model (1,1), the conditional average can be calculated from the following relation:

$$\hat{y}_{t+h|t} = E_t(y_{t+h}) = \hat{\mu}_{t+h|t} + \psi_1(\hat{y}_{t+h-1|t} - \hat{\mu}_{t+h|t}) + \Theta_1\hat{\varepsilon}_{t+h-1|t} \quad (42)$$

In the case of the ARFIMA model (0, d, 0), the relation used to calculate the conditional mean is:

$$\hat{y}_{t+h|t} = E_t(y_{t+h}) = \hat{\mu}_{t+h|t} + \sum_{i=1}^{\infty}\chi_i(\hat{y}_{t+h-i|t} - \hat{\mu}_{t+h|t}) \quad (43)$$

and in the case of the ARFIMA model (l, d, l) it results:

$$\hat{y}_{t+h|t} = E_t(y_{t+h}) = \left[\hat{\mu}_{t+h|t} + \sum_{i=1}^{\infty}\chi_i(\hat{y}_{t+h-i|t} - \hat{\mu}_{t+h|t})\right] + \hat{\psi}_1\left\{\hat{y}_{t+h-1|t} - \left[\hat{\mu}_{t+h|t} + \sum_{k=1}^{\infty}\chi_k(\hat{y}_{t+h-k-1|t} - \hat{\mu}_{t+h|t})\right]\right\} + \Theta_1\hat{\varepsilon}_{t+h-1|t} \quad (44)$$

For the *h*-period forecast of the conditional variance for the main GARCH models, the model-specific equations are presented below. Thus, in the case of the GARCH model (p, q), the equation is as follows:

$$\hat{\sigma}^2_{t+h|t} = E_t(\sigma^2_{t+h}) = \hat{w} + \sum_{i=1}^{q} \hat{\alpha}_i \varepsilon^2_{t+h-i|t} + \sum_{i=1}^{p} \hat{\beta}_i \hat{\sigma}^2_{t+h-i|t} \quad (45)$$

where: $\varepsilon^2_{t+i|t} = E_t(\varepsilon^2_{t+i}) = \sigma^2_{t+i|t}, i > 0$; $\varepsilon^2_{t+i|t} = E_t(\varepsilon^2_{t+i}) = \varepsilon^2_{t+i}, i \leq 0$; $\sigma^2_{t+i|t} = E_t(\sigma^2_{t+i}) = \sigma^2_{t+i}, i \leq 0$

In the case of the TARCH model (p, q), we have the relation:

$$\hat{\sigma}^2_{t+h|t} = E_t(\sigma^2_{t+h}) = \hat{w} + \sum_{i=1}^{q} (\hat{\alpha}_i \varepsilon^2_{t+h-i|t} + \hat{\gamma}_i d_{t+h-i|t} \varepsilon^2_{t+h-i|t}) + \sum_{i=1}^{p} \hat{\beta}_i \hat{\sigma}^2_{t+h-i|t} \quad (46)$$

where $d_{t+i|t} = E_t(d_{t+i}) = d_{t+i}, i \leq 0$; $d_{t+i|t} = E_t(d_{t+i}) = \frac{1}{2}, i > 0$ for symmetric distributions and $d_{t+i|t} = E_t(d_{t+i}) = \frac{1}{1+\xi^2}, i > 0$ for the asymmetric Student distribution (with asymmetry coefficient ξ).

In the case of the APARCH model (p, q), the conditional variance can be predicted starting from the equation:

$$\hat{\sigma}^{\delta}_{t+h|t} = E_t\left(\sigma^{\delta}_{t+h}\right) = \hat{w} + \sum_{i=1}^{q} \hat{\alpha}_i E_t\left[(|\varepsilon_{t+h-i}| - \hat{\gamma}_i \varepsilon_{t+h-i})^{\delta}\right] + \sum_{i=1}^{p} \hat{\beta}_i \hat{\sigma}^2_{t+h-i|t} \quad (47)$$

where $E_t\left[(|\varepsilon_{t+i}| - \hat{\gamma}_i \varepsilon_{t+i})^{\delta}\right] = (|\varepsilon_{t+i}| - \hat{\gamma}_i \varepsilon_{t+i})^{\delta}, i \leq 0$și $E_t\left[(|\varepsilon_{t+i}| - \hat{\gamma}_i \varepsilon_{t+i})^{\delta}\right] = k_i \hat{\sigma}^{\delta}_{t+i|t}, i > 0$, and k_i depends on the type of distribution analyzed.

Conclusions

From the article "Econometric model used in the optimization of the portfolio over several periods", made on the basis of an extensive study on the models that meet the proposed desideratum - high yield for considered periods - a series of conclusions both theoretical and practical. Thus, a first conclusion is that in order to capture the dynamics of conditional variance it is necessary to use a GARCH process, because if we introduce an ARCH process with a very large number of parameters, there are difficulties in estimating based on this model. A GARCH process is mainly used in finance because it can capture the dynamics of conditional variance by taking into account a small number of parameters.

Regarding the modeling of the conditional mean equation, ARMA processes were used, which were extended to ARMAX, ARIMA, ARFIMA models, thus capturing the long memory property through the considered operator, which in turn represents an infinite amount of lags. of the data series.

The models exposed, studied and analyzed can be used effectively in optimizing the asset portfolio, over different periods of time. The study can be extended and applied in a practical way by building the appropriate econometric model, depending on the identified factorial / resultant variables.

Applying and testing models in the study of the efficiency of invested asset portfolios is an important step especially when we consider different time periods.

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