BULETINUL Universităţii Petrol – Gaze din Ploieşti	Vol. LX No. 1/2008	55 - 62	Seria Ştiinţe Economice

Using Probability to Measure the Insurance Company Risk

Cornelia Gaber*, Maria Stoica**

* Petroleum-Gas University of Ploieşti, Bd. Bucureşti 39, Ploieşti
e-mail: corneliagaber@yahoo.com

** "Nichita Stanescu" High School Ploiesti, str. Nalbei, nr.3, Ploieşti
e-mail: pmagdastoica@yahoo.com

Abstract

The insurance companies must create a minimal reserve based on the actuarial calculus, for each category of insured goods. This article presents the probability that the insurance company will not go bankrupt after successive compensation and use the smallest bankruptcy probability, so that the difference between the paid compensations or even the total request of compensation and the bonus received will not exceed the fund of the risk reserve.

Key words: *insurance, insurance bonus, insurance contracts, risk fund, probability, Poisson process*

The insurance contracts are signed between the two parts which assume mutual responsibilities. These contracts are generated by general contract conditions specific to any kind of insurance contract and by the contract conditions specific to the insurances and supplementary options attached to the basic insurance contract.

The contracting party is a person (natural or legal) who makes the insurance contract and pays a fixed sum of money called insurance bonus only once when the contract is made or in installments throughout its duration.

The insurer, the company which takes over the risk is obligated to pay the compensations generated by the risk affecting the insurer throughout the whole period of the insurance contract.

The event insured is the event at whose production the insurer is obligated by contract to pay the insurance compensation (the sum paid by the insurer when producing the event insured according to the contract).

The insured sum is the sum estimated in the insurance contract guaranteed by the insurer.

Each insurance company possesses a risk reserve, made for the exclusive purpose of covering the insurance compensations due to the risks during the insured period in order to avoid the state of getting ruined.

So that a certain risk could be insurable it must follow some conditions:

- the contracting party must be interested in the insurance product offered by the insurer;
- the risk must be authenticated according to normal terms;

- the probability for the risk to occur must be minimum;
- to reduce the spreading the risks must be well grouped;
- the risk must not be produced by the insured person.

Further on we suggest calculating the possibility that after successive compensations, the insurer would not get ruined and use a ruining possibility as small as possible that is the difference between the paid compensations or even the total request of compensation and the bonus received must not exceed the fund of the risk reserve.

The number of the compensation demands is not affected by their financial value.

The financial values of the compensations do not reciprocally affect and do not change themselves for a period of time.

To describe this complex problem in real cases there are to be considered:

- $N(t)$ - number of the compensation demands recorded during the time after which the insurer must pay the insurance indemnity;
- $Y_1, Y_2, \ldots$ - financial value due to each compensation demand separately;
- v - income from the insurance bonus made by the insurer;
- c - initial capital;

$S(t) = c + vt - \sum_{i-1}^{N(t)} Y_i$ u. m. Is the sum the company possesses at moment t.

May $N(t)$ = number of the compensation demands recorded during the period [0,*t*]; statistically speaking we consider $N(t)$ evolving according to a Poisson flux . $N(t)$ variable of a process Poisson $N = (N(t))_{t \in R_+}$ and $N(0) = 0$, the process diagram is a non decreasing ladder function with unitary growths and the conditions the Poisson process verifies are:

1) if $0 < t_1 < \ldots < t_n$, then $N(t_i) - N(t_{i-1}), i = \overline{1,n}\ n \in N$ are independent aleatory variables;

2) repartition of the aleatory variable $N(t+h) - N(t)$ depends only on *h*;

3) $p(h) = P(N(h) \geq 1) = \lambda h + o(h),\ h \to 0$ cu $\lambda > 0$;

4) $P(N(h) \geq 2) = o(h),\ h \to 0$;

Using the conditions 1) – 4) we obtain the repartition of the aleatory variable *X*(*t*)

We note $p_m(t) P(N(t) = m),\ m \in N,\ t \in R_+$.

According to condition 4) we get $\sum_{m \geq 2} p_m(t) = o(h)$, $h \to 0$.

As $p(h) = P(N(h) \geq 1) = p_1(h) + p_2(h) + \ldots$ then according to conditions 1) and 2) we determine:

$$p_0(t+h) = p_0(t) p_0(h) = p_0(t)(1 - p(h)) \text{ from which}$$

$$\frac{p_0(t+h) - p_0(t)}{h} = -p_0(t) \frac{p(h)}{h} = -p_0(t)(\lambda + \frac{o(h)}{h})$$

$$\lim_{h\to 0}\frac{p_0(t+h)-p_0(t)}{h}=-p_0(t)(\lambda+\lim_{h\to 0}\frac{0(h)}{h})$$

$p_0^{'}(t)=-\lambda p_0(t)$. The solution to this separable differential equation is $\frac{p_0'(t)}{p_0(t)}=-\lambda$,

results $\ln p_0(t)=-\lambda t+\ln c$.

$p_0(t)=c\mathrm{e}^{-\lambda t}$, $t\in R_+$, c constant that is determined using the initial condition $p_0(0)=1$, but $p_0(0)=c\mathrm{e}^0$ so $c\mathrm{e}^0=1$ results $c=1$ and the solution is $p_0(t)=\mathrm{e}^{-\lambda t}$, $t\in R_+$.

For m=1 we define $p_1(t+h)=p_1(t)p_0(h)+p_0(t)p_1(h)$.

For $m\geq 2$ we obtain

$$p_1(t+h)=p_1(t)p_0(h)+\sum_{i=1}^{m}p_{m-i}(t)p_i(h)=p_1(t)p_0(h)+p_{m-1}(t)p_1(h)+\sum_{i=1}^{m}p_{m-i}(t)p_i(h)$$

Conditions 3) and 4) allow the certainty of $p_0(h)$ thus:

$$p_0(h)=1-p(h)=1-\lambda h+o(h),$$

$$p_1(h)=p(h)+o(h)=\lambda h+o(h)\sum_{i=2}^{m}p_{m-i}(t)p_i(h)\leq\sum_{i=2}^{m}p_i(h)=o(h)$$

$$p_1(t+h)-p_1(t)=p_1(t)p_0(h)+p_0(t)p_1(h)-p_1(t)=p_1(t)\left[p_0(h)-1\right]+p_0(t)p_1(h)=$$

$$=\lambda hp_0(t)+o(h)-\lambda hp_1(t)$$

$$\frac{p_1(t+h)-p_1(t)}{h}=\lambda p_0(t)+\frac{o(h)}{h}-\lambda p_1(t)$$

$$p_1^{'}(t)=\lambda p_0(t)-\lambda p_1(t)$$

$$p_1^{'}(t)+\lambda p_1(t)=\lambda p_0(t)$$

$$p_1^{'}(t)+\lambda p_1(t)=\lambda\mathrm{e}^{-\lambda t} \tag{1}$$

$p_1^{'}(t)+\lambda p_1(t)=0$ results $\frac{p_1^{'}(t)}{p_1(t)}=-\lambda$ results $\ln p_1(t)=-\lambda t+\ln c_1$

$p_1(t)=c_1\,\mathrm{e}^{-\lambda t}$ is the solution to the homogeneous equation.

The solution to the non homogeneous equation is determined according to the pattern of the homogeneous one where $c_1=c_1(\ldots.)$

$p_{iq}(t)=c_1(t)\mathrm{e}^{-\lambda t}$ particular solution of the non homogeneous equation:

$$p_{1q}'(t)=c_1'(t)\,\mathrm{e}^{-\lambda t}-c_1(t)\,\mathrm{e}^{-\lambda t}$$

$$p_{1q}'(t)+\lambda p_{iq}(t)=t\lambda\,\mathrm{e}^{-\lambda t}$$

$$c_1'(t)\mathrm{e}^{-\lambda t} - \lambda c_1(t)\mathrm{e}^{-\lambda t} + \lambda c_1(t)\mathrm{e}^{-\lambda t} = \lambda\,\mathrm{e}^{-\lambda t}$$

$$c_1'(t) = \lambda \text{ results } c_1(t) = \lambda t + c_2$$

$$p_{iq}(t) = (\lambda t + c_2)\mathrm{e}^{-\lambda t}$$

The general solution of equation (1) is:

$$p_1(t) = p_{1o}(t) + p_{1p}(t)$$

$$p_1(t) = c_1\,\mathrm{e}^{-\lambda t} + (\lambda t + c_2)\mathrm{e}^{-\lambda t} \ , (\forall) t \in R_+$$

$$p_1(0) = 0 \text{ results } c_1 + c_2 = 0 \text{ results } c_2 = -c_1$$

$$p_1(t) = c_1\,\mathrm{e}^{-\lambda t} + \lambda t\mathrm{e}^{-\lambda t} - c_1\mathrm{e}^{-\lambda t}$$

$$p_1(t) = \lambda t\mathrm{e}^{-\lambda t}.$$

Therefore:

$$p_m(t+h) - p_m(t) = p_m(t)p_0(h) + p_{m-1}(t)p_1(h) + \sum_{i=2}^{m} p_{m-i}(t)p_i(h) - p_m(t) =$$
$$= p_m(t)\left[p_0(h) - 1\right] + p_{m-1}(t)p_1(h) + o(h) = -p_m(t)p(h) + p_m(t)\left[\lambda h + 2o(h)\right] + o(h) =$$
$$= -p_m(t)\left[\lambda h + o(h)\right] + \lambda h p_{m-1}(t) + o(h) = \lambda h\left[-p_m(t) + p_{m-1}(t)\right] + o(h)\left[-p_m(t) + 2p_{m-1}(t) + 1\right] : h$$

$$\frac{p_m(t+h) - p_m(t)}{h} = \lambda\left[-p_m(t) + p_{m-1}(t)\right] + \frac{o(h)}{h}\left[-p_m(t) + 2p_{m-1}(t) + 1\right]$$

$$p_m'(t) = -\lambda p_m(t) + \lambda p_{m-1}(t) \ , \ t \in R_+ \qquad (2)$$

There is made a change of function in relation (2):

$$p_m(t) = q(t)\mathrm{e}^{-\lambda t}$$

$$p_m'(t) = q_m'(t)\mathrm{e}^{-\lambda t} - \lambda q_m(t)\mathrm{e}^{-\lambda t}$$

$$p_{m-1}(t) = q_{m-1}(t)\mathrm{e}^{-\lambda t}$$

$$p_m(t) = \frac{(\lambda t)^m}{m!} e^{-\lambda t}, \ (\forall)\ m \in N^* \ , t \in R_+.$$

In conclusion $\forall\ t \in R_+$, the variable $N(t)$ has a law of small numbers of parameter λt, so $E(N(t)) = \lambda t$ and $D^2(N(t)) = \lambda t$ where λ represents the average of apparitions numbers for the compensation demands corresponding to the time unit.

We consider the values of the successive compensation a series of independent aleatory variables identically allotted to the distribution function G and noted $Y_1, Y_2, \ldots$.

The income from the insurance bonus made by the insurance company during the time unit is v, and the initial capital of the company is c, consequently the sum the company possesses at moment t is:

$$S(t) = c + vt - \sum_{i=1}^{N(t)} Y_i \text{ u.m.}$$

The purpose of the article is to determine the probability of the company not to bankrupt, that is $P\left(S(t) > 0,\ \forall t \in R_+\right)$.

We consider the function:

$$R(c) = P\left(S(t) > 0,\ \forall\, t \in R_+\right) \tag{3}$$

If X_1 is the moment for the arrival of the first compensation demand then:

$$R(c) = \lambda \int_0^\infty P(S(t) > 0, \forall t \in R_+ \mid X_1 = x) \mathrm{e}^{-\lambda x} \mathrm{d}x \tag{4}$$

So considering that the payment of the first compensation must be made we deduct that:

$$P\left(S(t) > 0,\ \forall t \in R_+ \middle| X_1 = x\right) = \int_0^\infty P(S(t) > 0, \forall t \in R_+ \mid X_1 = x, Y_1 = y) \mathrm{d}G(y) \tag{5}$$

and

$$P\left(S(t) > 0,\ \forall t \in R_+ \middle| X_1 = x, Y_1 = y\right) = R(c + vx - y) \tag{6}$$

and the function R tests the equation :

$$R(c) = \lambda \int_0^\infty \left[\int_0^{c+vx} R(c + vx - y) \mathrm{d}G(y) \right] \mathrm{e}^{-\lambda x} \mathrm{d}x \tag{7}$$

The variable change $c + vx = u$ turns $R(c)$ defined by relation (7) into

$$R(c) = \frac{\lambda}{v} \int_0^\infty \left[\int_0^u R(u - y) \mathrm{d}G(y) \right] \mathrm{e}^{-\lambda\left(\frac{u}{v} - \frac{c}{v}\right)} \mathrm{d}u$$

From which

$$R(c) \mathrm{e}^{-\lambda \frac{c}{v}} = \frac{\lambda}{v} \int_c^\infty \left[\int_0^u R(u - y) \mathrm{d}G(y) \right] \mathrm{e}^{-\lambda \frac{u}{v}} \mathrm{d}u \tag{8}$$

Relation (8) by differentiation becomes:

$$\mathrm{e}^{-\lambda \frac{c}{v}} \left[R'(c) - \frac{\lambda}{v} R(c) \right] = -\frac{\lambda}{v} \mathrm{e}^{-\lambda \frac{c}{v}} \int_0^u R(c - y) \mathrm{d}G(y)$$

from which we get:

$$R'(c) - \frac{\lambda}{v} R(c) = -\frac{\lambda}{v} \int_0^u R(c - y) \mathrm{d}G(y)$$

That is:

$$R'(c)=\frac{\lambda}{v}R(c)-\frac{\lambda}{v}\int_0^u R(c-y)\mathrm{d}G(y) \tag{9}$$

For $c\in[0\,,w]$ relation (9) by integration defines $R(w)$ thus:

$$R(w)=R(0)+\frac{\lambda}{v}\int_0^w R(c)\mathrm{d}c-\frac{\lambda}{v}\int_0^w\left[\int_0^u R(c-y)\mathrm{d}G(y)\right]\mathrm{d}c$$

Which for $s=c-y$ and changing the integration regularity we get:

$$R(w)=R(0)+\frac{\lambda}{v}\int_0^w R(c)\mathrm{d}c-\frac{\lambda}{v}\int_0^u\left[\int_0^{w-s} R(s)\mathrm{d}s\right]\mathrm{d}G(y) \tag{10}$$

Let's consider $U(x)=\int_0^x R(s)\mathrm{d}s$ so $\int_0^{w-y} R(s)\mathrm{d}s=U(w-y)$ and from (10) using the previous substitutions and the side integration we get:

$$R(w)=R(0)+\frac{\lambda}{v}U(w)-\frac{\lambda}{v}\left\{U(w)-\int_0^w R(w-y)[1-G(y)]\mathrm{d}y\right\}$$

$$R(w)=R(0)+\int_0^w R(w-y)\frac{\lambda}{v}[1-G(y)]\mathrm{d}y \tag{11}$$

Equation (11) is a renewal equation having the density function $\frac{\lambda}{v}[1-G(y)]$. Knowing that the average for variable Y_1 is $EY_1=\mu$ and $\frac{\lambda}{v}\int_0^\infty[1-G(y)]\mathrm{d}y=\frac{\lambda\mu}{v}$ we consider that the density function is improper $\frac{\lambda}{v}[1-G(y)]$.

If $\frac{\lambda\mu}{v}>1$ or $\mu>\frac{v}{\lambda}$ then $R(c)=0$ because the compensation average per time unit exceeds income v.

If $\frac{\lambda\mu}{v}<1$ or $\mu<\frac{v}{\lambda}$ then $R(0)=1-\frac{\lambda\mu}{v}$ and $1-R(w)\sim\frac{1}{\alpha\mu^*}\left(1-\frac{\lambda\mu}{v}\right)\mathrm{e}^{\alpha w}$,

where $\mu^*=\frac{\lambda}{v}\int_0^\infty x\mathrm{e}^{\alpha x}[1-G(x)]\mathrm{d}x$, and α is determined from the equation: $\frac{\lambda}{v}\int_0^\infty \mathrm{e}^{\alpha x}[1-G(x)]\mathrm{d}x=1$. To establish a minimum reserve program it is considered that an insurance company makes contracts for goods of the same type, and the paid compensation for each commodity is represented by the aleatory variable with $M(Y_i)=m$ and $D^2(Y_i)=\sigma^2$, so the total compensation paid by the insurer is given by $Y=\sum_{i=}^{1} Y_i$, where $Y_1,...Y_k$ are the independent aleatory variables identically allotted.

Concerning the hypothesis that the clear bonus is equal to the average compensation we conclude that the total clear bonus is:

$$P_{net\ total} = M(Y) = km \tag{12}$$

And the risk fund *r* is defined by the inequality:

$$P(Y - P_{net\ total} > r) \leq \alpha,$$

or

$$P(Y - M(Y)) > r \leq \alpha,$$

from which we get

$$P\left(\frac{Y - M(Y)}{\sigma_{M(Y)}} > \frac{r}{\sigma_{M(Y)}}\right) \leq \alpha \tag{13}$$

If the number of the insured goods is big enough, then variable:

$$Z_k = \frac{Y - M(Y)}{\sigma_{M(Y)}} = \frac{Y - km}{\sigma\sqrt{k}} \in \mathrm{N}(0,1) \tag{14}$$

according to Lindeberg –Levy central limit theorem.

From (13) and (14) using the distribution function of the reduced normal distribution N(0,1)

$$\mathrm{F}(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \mathrm{d}t \tag{15}$$

We get:

$$\alpha \geq \mathrm{P}\left(\mathrm{Z}_k > \frac{r}{\sigma\sqrt{k}}\right) = 1 - \mathrm{P}\left(\mathrm{Z}_k \leq \frac{r}{\sigma\sqrt{k}}\right) = 1 - \mathrm{F}\left(\frac{r}{\sigma\sqrt{k}}\right) \tag{16}$$

Or $\mathrm{F}\left(\frac{r}{\sigma\sqrt{k}}\right) \geq 1 - \alpha = \mathrm{F}(Z_{1-\alpha})$.

Function F is non decreasing, thus $\frac{r}{\sigma\sqrt{k}} \geq Z_{1-\alpha}$, that is the minimum risk reserve $r_{\min} = \sigma Z_{1-\alpha}\sqrt{k}$.

Also using Cebâşev inequality:

$$P(|Y - M(Y)| < \varepsilon) \geq 1 - \frac{\sigma^2_{M(Y)}(Y)}{\varepsilon^2} \tag{17}$$

Considering $\varepsilon = t\sigma_{M(Y)} = r_1$ we get

$$P(|Y - M(Y)| < \sigma_{M(Y)}) \geq 1 - \frac{1}{t^2} \tag{18}$$

As $\{Y - M(Y) > r_1\} \subset \{|Y - M(Y)| \geq r_1\}$ it results:

$$P(Y - M(Y) > r_1) < P(|Y - M(Y)| \geq r_1) = 1 - P(|Y - M(Y)| < r_1) \leq 1 - \left(1 - \frac{1}{t^2}\right) = \frac{1}{t^2} \qquad (19)$$

Considering (19) $\alpha = t^{-2}$ we get $P(Y - M(Y) > r_1) \leq \alpha$, where α represents an accepted bankrupt probability , and from $\alpha = t^{-2}$ and $t\sigma_{M(Y)} = t\sigma = r_1$ we get $r_1 = \sigma\left(\sqrt{\alpha}\right)^{-1}$.

Considering that the average insured sum for each of the *k* insured goods is *s* and the probability of producing the calamity is *p* then variable $Y_i = (S\ \ p)^{\mathrm{T}}$ has $M(Y_i) = S\ p$ and $\sigma^2(Y_i) = M(Y_i^2) - M(Y_i) = S^2 p - S^2 p^2 = S^2 p(1-p) = S^2 pq,\ q = 1 - p$,

so $r_{\min} = \sqrt{S^2 pqk} Z_{1-k} = Z_{1-k} S\sqrt{pqk}$ while the risk reserve using Cebîşev inequity is $r_1 = S\sqrt{pqk}\left(\sqrt{\alpha}\right)^{-1}$ that is the bigger the bankrupt probability the lower the risk reserve and it is advisable that the insurance company should use a lower bankrupt probability even if a bigger reserve fund is necessary .

Conclusions

Establishing the insurance bonus amount and the risk fund is an activity on which the well functioning of any insurance company depends, so that a financial unbalance can occur and the company goes bankrupt.

The existence of several insurance companies creates a competition in the field of insurances and it is another aspect that must be taken into account by the insurance companies when they create their products.

The insurance companies that do not scientifically strengthen their bonus fees are in danger of having a small number of insurance demands due to the big price or a big number of insurance demands due to the low price, which eventually lead to high losses for the company.

References

1. M i h o c , Gh., C i u c u , G., C r a i u V. - *Teoria probabilităţilor şi statistică matematică*, Editura Didactică şi Pedagogică, Bucureşti, 1970
2. P u r c a r u , I. - *Matematică şi asigurări*, Editura Economică, Bucureşti, 1995
3. S t a n c u , S., M i h a i l , N. - *Decizii economice în condiţii de incertitudine*, Editura Economică, Bucureşti, 2005
4. *** http://www.casact.org/library/astin
5. *** http://www.xprimm.ro
6. *** http://www.1asig.ro

Măsurarea riscului unei societăţi de asigurare cu ajutorul probabilităţilor

Rezumat

Societăţile de asigurări, pentru fiecare categorie de bunuri asigurate, trebuie să-şi constituie o rezervă minimă fundamentată pe baza calculelor actuariale. În acest articol se prezintă probabilitatea ca după despăgubiri succesive, asigurătorul să nu se ruineze şi să folosească o probabilitate de ruinare cât mai mică, astfel încât diferenţa dintre despăgubirile achitate, sau chiar cererea totală de despăgubire şi primele încasate să nu depăşească fondul rezervei de risc.